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Hausdorff dimension of the set of nonergodic directions

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(with an Appendix by M. Boshernitzan)

Abstract

It is known that nonergodic directions in a rational billiard form a subset of the unit circle with Hausdorff dimension at most 1/2. Explicit examples realizing the dimension 1/2 are constructed using Diophantine numbers and continued fractions. A lower estimate on the number of primitive lattice points in certain subsets of the plane is used in the construction.

1. Introduction

Consider the billiard in a polygon Q. A fundamental result [KMS] implies that a typical trajectory with typical initial direction will be equidistributed provided the angles of Q are rational multiples of π. More precisely, there is a flat surface X associated to the polygon such that each direction $\theta \in S^1$ determines an area-preserving flow on X; the assertion is that the set $\mathrm{NE}(Q)$ of parameters θ for which the associated flow is not ergodic has measure zero. The statement holds more generally for the class of *rational billiards* in which the (abstract) polygon is assumed to have the property that the subgroup of $O(2)$ generated by the linear parts of the reflections in the sides is finite. For a recent survey of rational billiards, see [MT].

Let $Q_\lambda, \lambda \in (0,1)$, be the polygon described informally as a 2-by-1 rectangle with an interior wall extending orthogonally from the midpoint of a longer side so that its distance from the opposite side is exactly λ (see Figure 1). We are interested in the Hausdorff dimension of the set $\mathrm{NE}(Q_\lambda)$. Recall that λ is *Diophantine* if the inequality

$$\left|\lambda - \frac{p}{q}\right| \leqslant \frac{1}{|q|^e}$$

has (at most) finitely many integer solutions for some exponent $e > 0$.

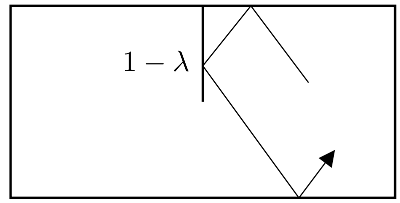

Figure 1. The billiard in Q_λ.

THEOREM 1. *If λ is Diophantine, then* $\mathrm{H.dim}\,\mathrm{NE}(Q_\lambda) = 1/2$.

In fact, Masur has shown that for any rational billiard the set of nonergodic directions has Hausdorff dimension at most 1/2 [Ma]. This upperbound is sharp, as Theorem 1 shows. It should be pointed out that the theorems in [KMS] and [Ma] are stated for holomorphic quadratic differentials on compact Riemann surfaces. The flat structure on the surface associated to a rational billiard is a special case, namely the square of a holomorphic 1-form.

The ergodic theory of the billiards Q_λ was first studied by Veech [V1] in the context of $\mathbb{Z}_2$ skew products of irrational rotations. Veech proved the slope of the initial direction θ has bounded partial quotients if and only if the corresponding flow is (uniquely) ergodic for all λ. On the other hand, if θ has unbounded partial quotients, then there exists an uncountable set $K(\theta)$ of λ for which the flow is not ergodic. In this way, Veech showed that minimality does not imply (unique) ergodicity for these $\mathbb{Z}_2$ skew products. (The first examples of minimal but uniquely ergodic systems had been constructed by Furstenberg in [Fu].) Our approach is dual to that of Veech in the sense that we fix λ and study the set of paramaters $\theta \in \mathrm{NE}(Q_\lambda)$.

The billiards Q_λ were first introduced by Masur and Smillie to give a geometric representation of the $\mathbb{Z}_2$ skew products studied by Veech. It follows from [V1] that $\mathrm{NE}(Q_\lambda)$ is countable if λ is rational. A proof of the converse can be found in the survey article [MT, Thm. 3.2]. Boshernitzan has given a short argument showing $\mathrm{H.dim}\,\mathrm{NE}(Q_\lambda) = 0$ for a residual (hence, uncountable) set of λ. (His argument is presented in the appendix to this paper.) Theorem 1 implies any such λ is a Liouville number. As is well-known, the set of Liouville numbers has measure zero (in fact, Hausdorff dimension zero). We remark that by Roth's theorem every algebraic integer satisfies the hypothesis of Theorem 1.

Some generalizations of Theorem 1 are mentioned in Section 2. For the class of Veech billiards (see [V2]) the set of nonergodic directions is countable. It would be interesting to know if there are (number-theoretic) conditions on a general rational billiard Q which imply that the Hausdorff dimension of $\mathrm{NE}(Q) = 1/2$.

Theorem 1 can be reduced to a purely number-theoretic statement.

LEMMA 1.1 (Summable cross products condition). *Suppose* (w_j) *is a sequence of vectors of the form* $(\lambda + m_j, n_j)$, *where* $m_j, n_j \in 2\mathbb{Z}$ *and* $n_j \neq 0$, *and assume that the Euclidean lengths* $|w_j|$ *are increasing. The condition*

$$\sum |w_j \times w_{j+1}| < \infty, \tag{1}$$

implies that $\theta_j = w_j/|w_j|$ *converges to some* $\theta \in \mathrm{NE}(Q_\lambda^t)$ *as* $j \to \infty$. (*Here,* Q_λ^t *is the billiard table obtained by reflecting* Q_λ *in a line of slope* -1.)

THEOREM 2. *Let* $K(\lambda)$ *be the set of nonergodic directions that can be obtain using Lemma* 1.1. *If* λ *is Diophantine*, *then* $\mathrm{H.dim}\, K(\lambda) = 1/2$.

Proof of Theorem 1. Theorem 2 implies $\mathrm{H.dim}\,\mathrm{NE}(Q_\lambda) = \mathrm{H.dim}\,\mathrm{NE}(Q_\lambda^t) \geqslant 1/2$. Together with Masur's upperbound, this gives Theorem 1. □

Density of primitive lattice points. The main obstacle in our approach to finding lowerbounds on Hausdorff dimension is the absence of *primitive* lattice points in certain regions of the plane. More precisely, let $\Sigma = \Sigma(\alpha, R, Q)$ denote the parallelogram (Figure 2)

$$\Sigma := \Big\{(x,y) \in \mathbb{R}^2 : |y\alpha - x| \leqslant 1/Q,\ R \leqslant y \leqslant 2R\Big\}$$

and define

$$\mathrm{dens}(\Sigma) := \frac{\#\{(p,q) \in \Sigma : \gcd(p,q) = 1\}}{\mathrm{area}(\Sigma)}.$$

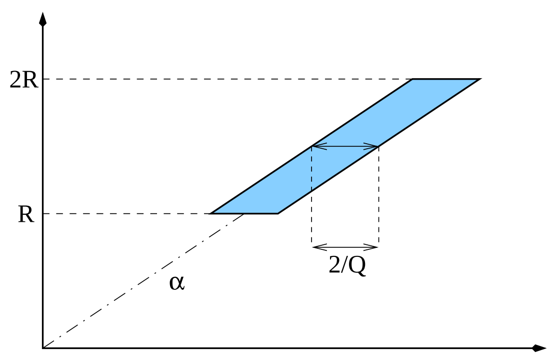

Figure 2. The parallelogram $\Sigma(\alpha, R, Q)$.

The proof of Theorem 2 relies on the following fact:

THEOREM 3. *Let* $\mathrm{Spec}(\alpha)$ *be the sequence of heights formed by the convergents of* α. *There exist constants* A_0 *and* $\rho_0 > 0$ *such that whenever* $\mathrm{area}(\Sigma) \geqslant A_0$

$$\mathrm{Spec}(\alpha) \cap [Q, R] \neq \emptyset \quad \Rightarrow \quad \mathrm{dens}(\Sigma) \geqslant \rho_0.$$

Remark. It can be shown $\mathrm{dens}(\Sigma) = 0$ if α does not have any convergent whose height is between $Q/4$ and $8R$. Thus, $\mathrm{area}(\Sigma) \gg 1$ alone cannot imply the existence of a primitive lattice point in Σ. For example, the implication

$$|\alpha| < \frac{1}{2R}\left(1 - \frac{1}{Q}\right) \quad \Rightarrow \quad \mathrm{dens}(\Sigma) = 0$$

is easy to verify and remains valid even if $|\cdot|$ is replaced by the distance to the nearest integer (because arithmetic density is preserved under $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$).

Outline. Theorem 2 is proved by showing that $K(\lambda)$ contains a Cantor set whose Hausdorff dimension may be chosen close to $1/2$ when λ is Diophantine. The construction of this Cantor set is based on Lemma 1.1 and is presented in Section 2. The proof of Theorem 2 is completed in Section 3 if we assume the statement of Theorem 3, whose proof is deferred to Section 4.

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2. Cantor set of nonergodic directions

We begin with the proof of Lemma 1.1, which is the recipe for the construction of a Cantor set $E(\lambda) \subset K(\lambda)$. We then show that the Hausdorff dimension of $E(\lambda)$ can be chosen arbitrarily close to $1/2$ if the arithmetic density of the parallelograms $\Sigma(\alpha, R, Q)$ can be bounded uniformly away from zero.

2.1. *Partition determined by a slit.* The flat surface associated to Q_λ is shown in Figure 3. It will be slightly more convenient to work with the reflected table Q_λ^t. Let X_λ be the flat surface associated to Q_λ^t. The proof of Lemma 1.1 is based on the following observation:

$$X_\lambda \text{ is a branched double cover of the square torus } T = \mathbb{R}^2/\mathbb{Z}^2.$$

More specifically, let $w_0 \subset T$ denote the projection of the interval $[0, \lambda]$ contained in the x-axis. X_λ may be realized (up to a scale factor of 2) by gluing two copies of the slit torus $T \setminus w_0$ along their boundaries so that the upper edge of the slit in one copy is attached to the lower edge of the slit in the other, and similarly for the remaining edges. The induced map $\pi : X_\lambda \to T$ is the branched double cover obtained by making a cut along the slit w_0.

LEMMA 2.1 (Slit directions are nonergodic). *A vector of the form $(\lambda + m, n)$ with $m, n \in 2\mathbb{Z}$ and $n \neq 0$ determines a nonergodic direction in Q_λ^t.*

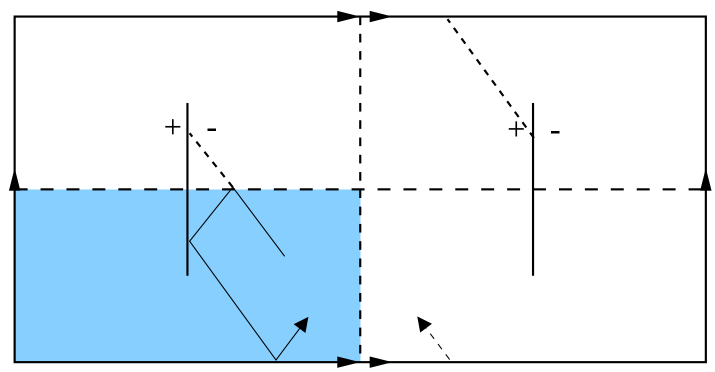

Figure 3. Unfolded billiard trajectory.

Proof. A vector of the given form determines a slit w in T that is homologous to w_0 (mod 2). (We assume λ is irrational, for the statement of the lemma is easily seen to hold otherwise.) If $\pi' : X' \to T$ is the branched double cover obtained by making a cut along w, then there is a biholomorphic isomorphism $h : X_\lambda \to X'$ such that $\pi = \pi' \circ h$. It follows that $\pi^{-1}(w)$ partitions X_λ into a pair of slit tori with equal area, and that this partition is invariant under the flow in the direction of the slit. Hence, the vector (after normalization) determines a nonergodic direction in Q^t_λ. □

Proof of Lemma 1.1. It is easy to see from (1) that the directions θ_j form a Cauchy sequence. The corresponding partitions of X_λ also converge in a measure-theoretic sense: the symmetric difference of consecutive partitions is a union of parallelograms whose total area is bounded by the corresponding term in (1); summability implies the existence of a limit partition. Invariance of the limit partition under the flow in the direction of θ will follow by showing that h_j, the component of w_j perpendicular to θ, tends to zero as $j \to \infty$ ([MS, Th. 2.1]). To see this, observe that the area of the right triangle formed by w_j and θ is roughly h_j times the Euclidean length of w_j; it is bounded by the tail in (1) and therefore tends to zero. (We have implicitly assumed that λ is irrational. For rational λ the lemma still holds because a nonzero term in (1) must be at least the reciprocal of the height.) □

Remark. A vector of the form $(\lambda + m, n)$ with $m, n \in g\mathbb{Z}$ and $n \neq 0$ determines a partition of the branched g-cyclic cover of T into g slit tori of equal area. From this, it is not hard to show that the conclusion of Theorem 1 holds in genus $g \geqslant 2$. Gutkin has pointed out other higher genus examples obtained by considering branched double covers along multiple parallel slits. Further examples are possible by observing that the proof of Theorem 2 depends only on a Diophantine condition on the vector $w_0 = (\lambda, 0)$. (See §3.)

2.2. *Definition of* $E(\lambda)$. Our goal is to find sequences that satisfy condition (1) and intuitively, the more we find, the larger the dimension. However, in order to facilitate the computation of Hausdorff dimension, we shall restrict our attention to sequences whose Euclidean lengths grow at some fixed rate.

We shall realize $E(\lambda)$ as a decreasing intersection of compact sets E_j, each of which is a disjoint union of closed intervals. Let V denote the set of vectors that satisfy the hypothesis of Lemma 2.1. Henceforth, by a *slit* we mean a vector $w \in V$ whose *length* is given by $L := |n|$ and *slope* by $\alpha := (\lambda + m)/n$. Note that the following version of the cross product formula holds: $|w \times w'| = LL'\Delta$, where Δ is the distance between the slopes. Fix a parameter $\delta > 0$.

Definition 2.2 (Children of a slit). Let w be a slit of length L and slope α. A slit w' is said to be a *child* of w if

(i) $w' = w + 2(p, q)$ for some relatively prime integers p and q

(ii) $|q\alpha - p| \leqslant 1/L \log L$ and $q \in [L^{1+\delta}, 2L^{1+\delta}]$.

LEMMA 2.3 (Chains have nonergodic limit). *The direction of* w_j *converges to a point in* $K(\lambda)$ *as* $j \to \infty$ *provided* w_{j+1} *is a child of* w_j *for every* j.

Proof. The inequality in (ii) (equivalent to $|w \times w'| \leqslant 1/\log L$) implies that the directions of the slits are close to one another. Hence, their Euclidean lengths are increasing since the length of a child is approximately $L^{1+\delta}$. The sum in (1) is dominated by a geometric series of ratio $1/(1+\delta)$. ☐

Choose a slit w_0 and call it the slit of level 0. The slits of level $j+1$ are defined to be children of slits of level j. Let $V' := \cup V'_j$ where V'_j denotes the collection of slits that belong to level j. Associate to each $w \in V'$ the smallest closed interval containing all the limits obtainable by applying Lemma 2.3 to a sequence beginning with w. Define $E(\lambda) := \cap E_j$ where E_j is the union of the intervals associated to slits in V'_j. It is easily seen that the diameters of intervals in E_j tend to zero as $j \to \infty$. Hence, every point of E arises as the limit obtained by an application of Lemma 2.3. Therefore, $E(\lambda) \subset K(\lambda)$.

2.3. *Computation of Hausdorff dimension.* We first give a heuristic calculation which shows that the Hausdorff dimension of $K(\lambda)$ is at most $1/2$. (This fact is not used in the proof of Theorem 1.) We then show rigorously that the Hausdorff dimension of $E(\lambda)$ is at least $1/2$ under a critical assumption: *each slit in* V' *has enough children.*

Recall the construction of the Cantor middle-third set. At each stage of the induction, intervals of length Δ are replaced with $m = 2$ equally spaced subintervals of common length Δ'. In this case, the Hausdorff dimension is exactly $\log 2/\log 3$, or $\log m/\log(1/\varepsilon)$ where $\varepsilon := \Delta'/\Delta = 1/3$.

For $K(\lambda)$ it is enough to consider sequences for which every term in (1) is bounded above. Associated to each slit of length L is an interval of length $\Delta = 1/L^2$. The number of slits of length approximately L' is at most $m = L'/L$. Their intervals have approximate length $\Delta' = 1/(L')^2$. Therefore,

$$\mathrm{H.dim}\, K(\lambda) \leqslant \frac{\log m}{\log(\Delta/\Delta')} = \frac{1}{2}.$$

To get a lowerbound on the Hausdorff dimension of $E(\lambda)$ we need to show there are lots of children and wide gaps between them. The number of children is exactly $2L^\delta/\log L$ times the arithmetic density of the parallelogram $\Sigma(\alpha, R, Q)$ where $R = L^{1+\delta}$ and $Q = L \log L$.

Lemma 2.4 (Slopes of children are far apart). *The slopes of any two children of a slit with length L are separated by a distance of at least $O(1/L^{2+2\delta})$.*

Proof. Let w be a slit of length L. A child w' has the form $w' = w + 2v$ for some $v = (p, q)$. If $w'' = w + 2v'$ is another child with $v' = (p', q')$, then $v' \neq v$. Since both pairs are relatively prime, $|p/q - p'/q'| \geqslant 1/qq' \geqslant 1/4L^{2+2\delta}$. The lemma follows by observing that the slope of w' satisfies

$$\left|\alpha' - \frac{p}{q}\right| = \frac{|w' \times v|}{L'q} = \frac{|w \times v|}{(L+2q)q} \leqslant \frac{L|q\alpha - p|}{2q^2} \leqslant \frac{1}{2L^{2+2\delta}\log L}. \qquad \square$$

Proposition 2.5 (Enough children implies dimension 1/2). *Suppose there exists $c_1 > 0$ such that every slit in V' has at least $c_1 L^\delta/\log L$ children in V', where L denotes the length of the slit. Then* $\mathrm{H.dim}\, K(\lambda) = 1/2$.

Proof. The length of a slit in V'_j is roughly $L_j = L_0^{(1+\delta)^j}$, where L_0 denotes the length of the initial slit w_0. The number of children is at least $m_j = c_1 L_j^\delta/\log L_j$ and their slopes are at least $\varepsilon_j = 1/4L_j^{2+2\delta}$ apart. It follows by well-known estimates for computing Hausdorff dimension (we use [Fa, Ex. 4.6]) that

$$\mathrm{H.dim}\, E(\lambda) \geqslant \liminf_{j\to\infty} \frac{\log(m_0 \cdots m_{j-1})}{-\log(m_j\varepsilon_j)} = \liminf_{j\to\infty} \frac{\sum_{i=0}^{j-1} \delta \log L_i}{(2+\delta)\log L_j} = \frac{1}{2+\delta}.$$

Together with the upperbound on $K(\lambda)$, this proves the lemma. $\square$

Remark. Theorem 3 allows us to determine when a slit has enough children. It should by pointed out that Diophantine λ does *not* imply every slit will have enough children. We shall show that Proposition 2.5 holds if V' is replaced by a suitable subset. (By the remark following Theorem 3 one can easily show there are slits that do not have any children and whose directions form a dense set.)

3. Diophantine condition

Let w_0 be the initial slit in the definition of $E(\lambda)$. The hypothesis that λ is Diophantine implies there are constants $e_0 > 0$ and $c_0 > 0$ such that

$$||w_0 \times v|| = \min_{n \in \mathbb{Z}} |w_0 \times v - n| \geqslant \frac{c_0}{|v|^{e_0}} \quad \text{for all } v \in \mathbb{Z}^2,\ v \neq 0.$$

Fix a real number N so that $e_0 < N\delta$. We assume the length of w_0 is at least some predetermined value $L_0 = L_0(\lambda, \delta, N, e_0, c_0)$.

Definition 3.1 (Normal slits). A slit of length L and slope α is said to be *normal* if for every real number $n, 1 \leqslant n \leqslant N+1$,

$$\operatorname{Spec}(\alpha) \cap [e^{n\delta} L \log L, L^{1+n\delta}] \neq \emptyset.$$

Let V'' be the subset of V' formed by normal slits of length $\geqslant L_0$.

PROPOSITION 3.2 (Normal slits have enough children). *There exists $c_1 > 0$ such that every slit in V'' has at least $c_1 L^\delta / \log L$ children in V''.*

To complete the proof of Theorem 2 we also need

LEMMA 3.3 (Normal slits exist). *Arbitrarily long normal slits exist.*

Proof of Theorem 2 *assuming Lemma* 3.3 *and Proposition* 3.2. We may choose the initial slit w_0 to lie in V'', which is nonempty by the lemma. The calculation in the proof of Proposition 2.5 applies to a subset of $E(\lambda)$ to give the same conclusion; in other words, the proposition implies $\mathrm{H.dim}\, K(\lambda) = 1/2$. □

We recall two classical results from the theory of continued fractions. The k^{th} convergent p_k/q_k of a real number α is a (reduced) fraction such that

$$\frac{1}{q_k(q_{k+1}+q_k)} \leqslant \left|\alpha - \frac{p_k}{q_k}\right| \leqslant \frac{1}{q_k q_{k+1}} \tag{2}$$

and satisfies the recurrence relation $q_{k+1} = a_{k+1} q_k + q_{k-1}$ (similarly for p_k), where a_k is the k^{th} partial quotient. A partial converse is that if p and $q > 0$ are integers satisfying

$$\left|\alpha - \frac{p}{q}\right| \leqslant \frac{1}{2q^2} \tag{3}$$

then p/q is a convergent of α, although it need not be reduced.

3.1. *Existence of normal slits.*

Definition 3.4. A slit of length L and slope α is said to be *n-good* if

$$\operatorname{Spec}(\alpha) \cap [e^{n\delta} L \log L, L^{1+\delta}] \neq \emptyset.$$

LEMMA 3.5. *A sufficiently long N-good slit is normal.*

Proof. An $(N+1)$-good slit is normal by definition, so it suffices to consider the case of an N-good slit that is not $(N+1)$-good. Suppose w is such a slit, with length L and slope α. Let q_k be the largest height in $\mathrm{Spec}(\alpha) \cap [1, L^{1+\delta}]$ so that $q_k = e^{n_1 \delta} L \log L$ for some n_1 between N and $N+1$. Set $v := (p_k, q_k)$. By the RHS of (2), the Diophantine condition, $|v| \in O(L \log L)$ and $e_0 < N\delta$ we get

$$q_{k+1} \leqslant |q_k \alpha - p_k|^{-1} \leqslant (1/c_0) L |v|^{e_0} \leqslant L^{1+N\delta}$$

provided $L \geqslant L_0$. Since $N \leqslant n_1$, this shows w is normal. □

Proof of Lemma 3.3. By the previous lemma, it is enough to prove the existence of arbitrarily long N-good slits. We show that a sufficiently long slit that is not N-good has a nearby slit that is N-good.

Hence, let w be a slit of length L and slope α and assume it is not N-good. Let q_k be the largest height in $\mathrm{Spec}(\alpha) \cap [1, L^{1+\delta}]$. Since $q_{k+1} > L^{1+\delta}$ (here we use the irrationality of λ to guarantee the existence of the next convergent) the RHS of (2) implies $\Delta := (L|q_k\alpha - p_k|)^{-1} > L^{\delta}$. With $L' := L + 2mq_k$, it is not hard to see that there exists a positive integer m satisfying

$$e^{N\delta} \log(L') + 1/2 \leqslant \Delta \leqslant (L')^{\delta}.$$

Indeed, if m is smallest for the RHS, then the LHS holds when $L \geqslant L_0$.

Let $w' = w + 2mv$ where $v = (p_k, q_k)$. We show w' is N-good. Let α' be its slope. Using $|w' \times v| = |w \times v|$ and the cross product formula, we find $|\alpha' - p_k/q_k| = 1/L' q_k \Delta \leqslant 1/2q_k^2$ which by (3) implies $q_k \in \mathrm{Spec}(\alpha')$. Using the above inequalities on Δ in parallel with those in (2) we obtain

$$\begin{aligned} q_{k+1} &\leqslant L'\Delta \leqslant (L')^{1+\delta} \quad \text{and} \\ q_{k+1} &\geqslant L'(\Delta - q_k/L') \geqslant e^{N\delta} L' \log L' \end{aligned}$$

which show that w' is N-good. □

3.2. *Normal slits have enough normal children.* Assume w is a normal slit of length $L \geqslant L_0$ and slope α. Let q_k be the largest in $\mathrm{Spec}(\alpha) \cap [1, L^{1+\delta}]$ and define $n_1 \geqslant 1$ uniquely by $q_k = e^{n_1\delta} L \log L$.

LEMMA 3.6 (Enough children). *Since w has at least $O(L^{\delta}/\log L)$ $(n-1)$-good children where $n := \min(n_1, N+1)$, if w' is a child with length L' and slope α', then $w' = w + 2(p_{k'}, q_{k'})$ and $q_{k'+1} \in [L' \log L', (L')^{1+\delta}]$ for some $q_{k'} \in \mathrm{Spec}(\alpha')$.*

LEMMA 3.7 (Most children are normal). *The number of children constructed in the previous lemma that are not normal is at most $O(L^{\delta-\delta^2} \log L)$.*

Proof of Proposition 3.2 *assuming the above lemmas.* A slit $w \in V''$ has enough normal children. These are all longer than L_0 and therefore lie in V''. $\square$

Proof of Lemma 3.6. Applying Theorem 3 with $Q = e^{n\delta} L \log L$ and $R = L^{1+\delta}$ gives $O(L^\delta / \log L)$ children of the form $w' = w + 2(p, q)$ where $\gcd(p, q) = 1$, $q \in [L^{1+\delta}, 2L^{1+\delta}]$ and $|q\alpha - p|^{-1} \geqslant e^{n\delta} L \log L$.

Observe that $L' = L + 2q$ so that $|\alpha' - p/q| = L|q\alpha - p|/L'q \leqslant 1/2q^2$. Since $\gcd(p, q) = 1$, (3) implies $q = q_{k'} \in \operatorname{Spec}(\alpha')$ for some index k'.

It remains to bound the next height $q_{k'+1} \in \operatorname{Spec}(\alpha')$. Using the LHS of (2) together with the lower bound on $|q\alpha - p|^{-1}$, we have

$$q_{k'+1} \geqslant L'/L|q\alpha - p| - q_{k'} \geqslant L'(e^{n\delta} \log L - 1/2) \geqslant e^{(n-1)\delta} L' \log L'$$

since $L \geqslant L_0$. Using the RHS of (2) and $L \geqslant L_0$ again

$$q_{k'+1} \leqslant L'/Lq|\alpha - p/q| \leqslant L' L^\delta \leqslant (L')^{1+\delta}.$$

(In the second step we used the fact that $|\alpha - p/q| \geqslant 1/L^{2+2\delta}$ which, by Lemma 2.4, holds for all the children with a finite number of exceptions.) $\square$

Proof of Lemma 3.7. Let $\tilde{V}$ be the collection of slits formed by the children constructed in Lemma 3.6 that are not normal. We show that $\tilde{V}$ has at most $O(L^{\delta - \delta^2} \log L)$ elements. Observe that if $n_1 \geqslant N + 1$, all children constructed are N-good, hence normal, by Lemma 3.5. In this case, $\tilde{V}$ is empty and we have nothing to prove. Therefore, we may assume $n_1 < N + 1$.

The next lemma will allow us to count the number of elements in $\tilde{V}$.

LEMMA 3.7. *Let w' be a slit in $\tilde{V}$ of length L' and slope α'. Then* (i) *the largest $q_{l'} \in \operatorname{Spec}(\alpha') \cap [1, (L')^{1+\delta}]$ lies in $[L' \log L', e^{N\delta} L' \log L']$ and* (ii) *it satisfies the inequality below for at most finitely many possible values of a.*

$$|L(q_{l'}\alpha - p_{l'}) \pm 2a| \leqslant \frac{1}{L^{n_2\delta + n_2\delta^2}} \quad \textit{where } n_2 := \max(1, n_1 - 1). \tag{4}$$

Proof. By definition $q_{l'+1} = (L')^{1+n'\delta}$ for some $n' \geqslant 1$. Since w' is $(n_1 - 1)$-good, $q_{l'} = e^{n''\delta} L' \log L'$ for some $n'' \geqslant n_1 - 1 \geqslant 0$. By Lemma 3.5, w' is not N-good so that $n'' < N$; this proves (i).

Since $n_1 < N + 1$, the fact that w' is not normal implies $n' \geqslant n'' \geqslant n_2$.

Let $q_{k'} \in \operatorname{Spec}(\alpha')$ be as in Lemma 3.6 and recall $q_{k'+1} \in [L' \log L', (L')^{1+\delta}]$. By definition of $q_{l'}$, $q_{l'} \geqslant q_{k'+1}$. The recurrence relations satisfied by convergents imply that $q_{l'} = a q_{k'+1} + b q_{k'}$ for some integers $a > 0$ and $b \geqslant 0$. By (i), $a \leqslant e^{N\delta}$.

Write $w' = w + 2v$ and note that $|w' \times v| = |w \times v|$. Since the cross product of consecutive convergents is ± 1 (thought of as vectors),

$$L'|q_{l'}\alpha' - p_{l'}| = |L(q_{l'}\alpha - p_{l'}) \pm 2a|.$$

The RHS of (2), $L' = L + 2q \geqslant L^{1+\delta}$, and $n' \geqslant n_2$ imply

$$L'|q_{l'}\alpha' - p_{l'}| \leqslant \frac{L'}{q_{l'+1}} \leqslant \frac{1}{(L^{1+\delta})^{n'\delta}} \leqslant \frac{1}{L^{n_2\delta + n_2\delta^2}}$$

and (ii) follows. □

Lemma 3.7 allows us to write $\tilde{V}$ as a finite union of subsets $\tilde{V}_{\pm a}$. Let $Q_{\pm a}$ denote the corresponding set of heights $q_{l'}$ associated to the slits in $\tilde{V}_{\pm a}$. The next two lemmas complete the proof of Lemma 3.7. □

LEMMA 3.8. *$\tilde{V}_{\pm a}$ and $Q_{\pm a}$ have the same number of elements.*

Proof. We need to show that the map $\tilde{V}_{\pm a} \to Q_{\pm a}$ sending w' to $q_{l'}$ is injective. Let w'' be different from w' with corresponding image $q_{l''}$. Note that since $|\alpha' - p_{l'}/q_{l'}| \leqslant 1/(L')^{1+\delta} q_{l'}$ is small compared to the distance between the slopes of w' and w'' (Lemma 2.4), the rationals $p_{l'}/q_{l'}$ and $p_{l''}/q_{l''}$ are distinct. Their heights differ because the interval containing them is smaller than $1/q_{l'}$. □

LEMMA 3.9. *Each $Q_{\pm a}$ is a union of at most $O(\log L)$ subsets, each having at most $O(L^{\delta-\delta^2})$ elements.*

Proof. Let $q_{l'}, q_{l''} \in Q_{\pm a}$ and set $\bar{q} := |q_{l''} - q_{l'}|$. We claim that if $\bar{q} \leqslant L^{1+\delta}$ then $\bar{q} = dq_k$ for some positive integer $d \in O(L^{\delta-\delta^2})$. This implies the lemma since we either have $\bar{q} > L^{1+\delta}$ or $\bar{q} \ll L^{1+\delta}$ so that the elements of $Q_{\pm a}$ fall into $O(\log L)$ clusters, each having $O(L^{\delta-\delta^2})$ elements.

Hence, assume $\bar{q} \leqslant L^{1+\delta}$ and set $\bar{p} = |p_l - p_{l'}|$. Let p/q be the reduced form of $\bar{p}/\bar{q}$ so that $\bar{q} = dq$ where $d = \gcd(\bar{p}, \bar{q})$. From (4) and the triangle inequality

$$|\bar{q}\alpha - \bar{p}| \leqslant \frac{2}{L^{1+n_2\delta+n_2\delta^2}}.$$

Since $q \leqslant \bar{q} \leqslant L^{1+\delta}$, $|\alpha - p/q| \leqslant 1/2q^2$. By (3) p/q is a convergent of α; since $\gcd(p,q) = 1$, $q = q_{k'} \in \mathrm{Spec}(\alpha)$ for some index k'. By hypothesis, $q \leqslant L^{1+\delta}$, so we must have $k' \leqslant k$. In fact, we must have $k' = k$ because $k' < k$ implies $|\bar{q}\alpha - \bar{p}| = d|q_{k'}\alpha - p_{k'}| \geqslant 1/(q_k + q_{k-1}) \geqslant 1/2q_k$ which contradicts the previous inequality. Using the LHS of (2), we now have

$$d = \frac{|\bar{q}\alpha - \bar{p}|}{|q_k\alpha - p_k|} \leqslant \frac{2(q_{k+1} + q_k)}{L^{1+n_2\delta+n_2\delta^2}}$$

which is $O(L^{(n_3-n_2)\delta-n_2\delta^2})$ where n_3 is defined by $q_{k+1} = L^{1+n_3\delta}$. Since w is normal, $n_3 \leqslant n_1$. This together with (4) implies $n_3 - n_2 \leqslant n_1 - n_2 \leqslant 1$ and $n_2 \geqslant 1$; thus proving the claim. $\square$

4. Counting rationals in intervals

Primitive lattice points in the parallelogram Σ correspond to rationals in the interval $I := \left[\alpha - \frac{1}{RQ}, \alpha + \frac{1}{RQ}\right]$. Set

$$\Lambda_I := \left\{(x,y) \in \mathbb{R}^2 : x/y \in I,\ R \leqslant y \leqslant 2R\right\}.$$

THEOREM 4. *If* $\mathrm{Spec}(\alpha) \cap [Q,R] \neq \emptyset$, $\mathrm{dens}(\Lambda_I) \geqslant 1/24$ *and* $R/Q \geqslant 16$.

Proof of Theorem 3. We use $A_0 = 16$ and $\rho_0 = 1/32$. Observe that Σ contains $\Lambda_{I'}$ where I' is concentric with I and half as wide. Moreover, $\Lambda_{I'}$ occupies three quarters of its area. Replacing Q with $2Q$ in Theorem 4 and assuming $\mathrm{Spec}(\alpha) \cap [2Q,R] \neq \emptyset$, we conclude $\mathrm{dens}(\Sigma) \geqslant (3/4)\,\mathrm{dens}(\Lambda_{I'}) \geqslant \rho_0$.

If $\mathrm{Spec}(\alpha) \cap [2Q,R] = \emptyset$, let q_k be the largest height in $\mathrm{Spec}(\alpha) \cap [Q,R]$. By the RHS of (2) it is easy to show the reduced fraction

$$\frac{p}{q} = \frac{ap_k + p_{k-1}}{aq_k + q_{k-1}} \quad \text{where} \quad a = 1, 2, 3, \ldots$$

(known as an intermediate fraction of α when $a \leqslant a_{k+1}$) satisfies

$$\left|\alpha - \frac{p}{q}\right| \leqslant \left(\frac{q_k + |q_{k+1} - q|}{q}\right) \frac{1}{q_k q_{k+1}}. \tag{5}$$

These correspond to R/q_k primitive lattice points in Σ, and since $q_k \leqslant 2Q$, it follows easily that $\mathrm{dens}(\Sigma) \geqslant 1/4 \geqslant \rho_0$. $\square$

LEMMA 4.1. *If J has rational endpoints of height at most R*, $\mathrm{dens}(\Lambda_J) \geqslant 1/6$.

Proof. We shall first prove the lemma under the additional hypotheses:

(i) the height of any rational in $\mathrm{int}(J)$ is greater than R, and

(ii) $|pq' - p'q| = 1$, where p/q and p'/q' are the endpoints of J.

By (ii), arithmetic density is preserved by the linear map γ which sends the standard basis to lattice points corresponding to the endpoints of J. Note that

$$\gamma^{-1}(\Lambda_J) \subset \Delta := \left\{(x,y) \in \mathbb{R}^2 : x/a + y/a' \leqslant 2,\ x, y > 0\right\}$$

where $a = R/q$ and $a' = R/q'$. Let $n(\Delta)$ denote the number of primitive lattice points $\mathrm{int}(\Delta)$. By (i), it is enough to show $n(\Delta) \geqslant aa'/4$.

Without loss of generality, assume $a' \geqslant a \geqslant 1$. There are two cases. If $a' \leqslant 2$ then $aa' \leqslant 4$, and since $(1,1) \in \Delta$ we have $n(\Delta) \geqslant 1 \geqslant aa'/4$. On the other hand, if $a' > 2$, then since $\gamma(1,1) \in \Lambda_J$, we have $1/a + 1/a' \geqslant 1$ so that $a \in [1,2]$. Considering pairs of the form $(1,n)$, we find

$$n(\Delta) \geqslant \left\lfloor a'\left(2 - \frac{1}{a}\right)\right\rfloor > aa'\left(\frac{2}{a} - \frac{1}{a^2} - \frac{1}{2}\right) \geqslant \frac{aa'}{4}.$$

This completes the proof assuming the additional hypotheses.

Note that every interval is a disjoint union of intervals satisfying (i). Hence, the lemma follows if we show (i) = (ii). Indeed, let $d = |pq' - p'q|$. There is a linear map in $\mathrm{GL}_2\mathbb{Z}$ that takes (p,q) to $(0,1)$ and (p',q') to (d,d') for some integer $d', 0 \leqslant d' < d$. If $d' > 0$ then $(1,1)$ is contained in the triangle determined by the origin, $(1,0)$ and (d,d') and corresponds to a rational of height at most R in $\mathrm{int}(J)$. Therefore, (i) implies $d' = 0$, and since $\gcd(d,d') = 1$, this in turn implies that $d = 1$, giving (ii). ☐

The height of a rational strictly between p/q and p'/q' is at least $q + q'$:

$$\frac{1}{qq'} = \left|\frac{p}{q} - \frac{p''}{q''}\right| + \left|\frac{p''}{q''} - \frac{p'}{q'}\right| \geqslant \frac{1}{qq''} + \frac{1}{q'q''}$$

This will be used several times in the next proof.

Proof of Theorem 4. Let α' and α'' denote the left and right endpoints of I, respectively. Let q_k be the largest height in $\mathrm{Spec}(\alpha) \cap [Q,R]$. By (2), $p_k/q_k \in I$ and without loss of generality, we assume $p_k/q_k \leqslant \alpha$. Let q_l be the largest height in $\mathrm{Spec}(\alpha'') \cap [1,R]$. Since p_k/q_k cannot lie strictly between p_l/q_l and p_{l+1}/q_{l+1}, it must be the case that $p_k/q_k \leqslant p_l/q_l$. In fact, strict inequality must hold because $q_l = q_k \geqslant Q$ and $|\alpha' - p_k/q_k| \leqslant 1/q_l q_{l+1} < 1/RQ$ give a contradiction.

Let $J := [p_k/q_k, p_l/q_l]$. We claim its length is at least $1/2RQ$. In fact, p_l/q_l lies within $1/2RQ$ of α'' if $q_l \geqslant 2Q$. On the other hand, $q_l \leqslant 2Q$ implies $|J| \geqslant 1/q_k q_l \geqslant 1/2RQ$. In either case, $|J| \geqslant 1/2RQ$.

We may assume $\alpha'' \leqslant p_l/q_l$, for otherwise $J \subset I$ and $\mathrm{dens}(\Lambda_I) \geqslant 1/24$.

There are three cases. First, if $q_{l+1} + 3q_l > 2R$ there can be at most two rationals with height at most $2R$ that lie strictly between p_l/q_l and p_{l+1}/q_{l+1}:

$$\frac{p_{l+1}}{q_{l+1}} < \frac{p_{l+1} + p_l}{q_{l+1} + q_l} < \frac{p_{l+1} + 2p_l}{q_{l+1} + 2q_l} < \frac{p_l}{q_l}.$$

Let $n(\Lambda_I)$ denote the number of primitive lattice points in Λ_I. By Lemma 4.1,

$$n(\Lambda_I) \geqslant \frac{\mathrm{area}(\Lambda_J)}{6} - 2 = \left(\frac{1}{12} - \frac{2Q}{3R}\right)\mathrm{area}(\Lambda_I)$$

and since $R/Q \geqslant 16$, $\mathrm{dens}(\Lambda_I) \geqslant 1/24$.

Next, suppose $q_{l+1}+3q_l \leqslant 2R$ and $q_l \geqslant 2Q$. Note that an intermediate fraction of α'' with height between q_l and q_{l+1} lies to the left of α''. Let $J' := [p_k/q_k, p/q]$ where p/q is the intermediate fraction with the largest height not exceeding R. By definition, $q > R - q_l$. From (5) we have

$$\left|\alpha - \frac{p}{q}\right| \leqslant \left(\frac{q_l + q_{l+1} - q}{q}\right)\frac{1}{q_l q_{l+1}} \leqslant \left(\frac{2R - 2q_l - q}{R - q_l}\right)\frac{1}{2RQ} \leqslant \frac{1}{2RQ}$$

so that $|J'| \geqslant 1/2RQ$ and $\mathrm{dens}(\Lambda_I) \geqslant 1/24$.

Finally, assume that $q_{l+1}+3q_l \leqslant 2R$ and $q_l < 2Q$. Again, we consider the intermediate fractions of α''. Observe that their heights are at most $2R$, since $q_{l+1} \leqslant 2R$. They form a sequence that increase towards α'' from the left. Given a consecutive pair with heights less than R we can always find a rational strictly in between them with height in $[R, 2R]$. It follows that the number of rationals in I with height in $[R, 2R]$ is at least $(q_{l+1} - q)/q_l$, where q is the height of the first intermediate fraction that falls into I. According to (5), an intermediate fraction lies in I as soon as its height is greater than

$$R' := \frac{(q_l + q_{l+1})RQ}{2q_l q_{l+1} + RQ}.$$

Hence, $q - q_l \leqslant R'$ and

$$\begin{aligned} n(\Lambda_I) &\geqslant \frac{q_{l+1} - q}{q_l} \geqslant \frac{q_{l+1} - R'}{q_l} - 1 = \frac{2q_{l+1}^2 - RQ}{2q_l q_{l+1} + RQ} - 1 \\ &\geqslant \frac{2R^2 - RQ}{9RQ} - 1 = \frac{2}{27}\left(1 - \frac{5Q}{R}\right)\mathrm{area}(\Lambda_I). \end{aligned}$$

Since $R/Q \geqslant 16$, we get $\mathrm{dens}(\Lambda_I) \geqslant 1/24$. □

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Appendix

By Michael Boshernitzan

With notation as in the introduction, for $\lambda \in [0, 1)$, set

$$h(\lambda) \stackrel{\text{def}}{=} \text{H.dim NE}(Q_\lambda). \tag{6}$$

Recall (see Introduction) that $h(\lambda) \leq 1/2$ for all λ [Ma], and, by Theorem 1, that $h(\lambda) = 1/2$ for all Diophantine λ. We also have $h(\lambda) = 0$ for rational λ (then the set Q_λ is in fact countable [V1]). The main result in this section is given by the following theorem.

THEOREM 5. *The set of* $\lambda \in [0,1)$ *for which* H.dim $\mathrm{NE}(Q_\lambda) = 0$ *form a residual subset of* $[0,1)$. *In particular, there are irrational* $\lambda \in [0,1)$ *such that* $h(\lambda) = 0$.

Recall that a subset $A \subset X$ of a topological space X is called *residual* (or topologically large) if it contains a dense G_δ-subset of X. A subset $Y \subset X$ is called a G_δ-set (in X) if Y is a countable intersection of open subsets of X. Its complement $X \setminus Y$ is called an F_σ-set.

We remark that no irrational number λ satisfying $h(\lambda) = 0$ is known even though the set is topologically large (in particular, uncountable). Any such λ must be Liouville. (Note that the set of Liouville numbers forms a residual set of Lebesgue measure 0.)

A.1. $\mathbb{Z}_2$ skew products of irrational rotations

Let $X = S_0^1 \cup S_1^1 = S^1 \times \{0,1\}$ be the union of two unit circles $S_k^1 = S^1 \times \{n\}$, $n \in \{0,1\}$, and consider the two-parameter family of transformations

$$\rho_{\alpha,\lambda} : X \to X, \qquad \alpha \in \mathbb{R},\ \lambda \in K = [0,1),$$

defined as follows. For $x = (s,n) \in X$,

$$\rho_{\alpha,\lambda}(x) = \rho_{\alpha,\lambda}(s,n) = (s \oplus \alpha, n'), \qquad n' = \begin{cases} n, & \text{if } 0 \le s < \lambda, \\ 1-n, & \text{if } \lambda \le s < 1, \end{cases} \tag{7}$$

where $x = (s,n) \in X$, $\quad s, s \oplus \alpha \in S^1 = \mathbb{R}/\mathbb{Z} = [0,1)$, and $\oplus$ stands for the group operation in S^1.

The dynamical systems $(X, \rho_{\alpha,\lambda})$ have been studied by Veech [V1] as particular $\mathbb{Z}_2$ skew product extensions of irrational α-rotations. Indeed, $\rho_{\alpha,\lambda}$ may be interpreted as the first return map to a disjoint union of two circles embedded in the surface associated to Q_λ. In particular, properties of billiards on Q_λ reduce to the study of dynamical systems $(X, \rho_{\alpha,\lambda})$. One verifies that the ergodicity of the billiard system Q_λ in direction θ is equivalent to the ergodicity of the map $\rho_{\alpha,\lambda}$ with $\alpha = \tan(\theta)$ (the slope in direction θ). Denote

$$\mathrm{NE}(X) = \{(\alpha,\lambda) \in \mathbb{R} \times [0,1) \mid (X, \rho_{\alpha,\lambda}) \text{ is not ergodic}\},$$

and, for $\lambda \in [0,1)$,

$$\begin{aligned} \mathrm{NE}(X_\lambda) &= \{\alpha \in \mathbb{R} \mid (\alpha,\lambda) \in \mathrm{NE}(X)\} \\ &= \{\alpha \in \mathbb{R} \mid (X, \rho_{\alpha,\lambda}) \text{ is not ergodic}\}. \end{aligned} \tag{8}$$

The sets $\mathrm{NE}(X_\lambda)$, $\mathrm{NE}(Q_\lambda)$ have the same Hausdorff dimensions because

$$\mathrm{NE}(X_\lambda) = \mathrm{NE}(X_\lambda) + 1 = \tan(\mathrm{NE}(Q_\lambda)) = \{\tan(\theta) \mid \theta \in \mathrm{NE}(Q_\lambda)\}.$$

Thus we have (see (6))

$$h(\lambda) = \text{H.dim } \mathrm{NE}(Q_\lambda) = \text{H.dim } \mathrm{NE}(X_\lambda). \tag{9}$$

A.2. The topological lemma

The following lemma is central in the proof of Theorem 5.

LEMMA A.1. *Let L be a G_δ-subset of a σ-compact metric space K. Let P be a Polish space* (*a space with a complete metric topology*). *Let H be an F_σ-subset of the cartesian product $W = K \times P$. For every $p \in P$, denote*

$$K(p) = \{k \in K \mid (k,p) \in H\} \subset K \tag{10}$$

and

$$P_o = \{p \in P \mid K(p) \subset L\}. \tag{11}$$

If P_o is dense in P, then P_o is a residual subset (*i.e., contains a dense G_δ-subset*) *of P. In particular, P_o is uncountable if P is.*

Proof. Since the family of residual subsets of P is closed under countable intersections, we assume (as we may without loss of generality) that K is compact and H is closed in $W = K \times P$.

Since P is separable and P_o is dense in P, there is a countable subset $P_c \subset P_o$ which is dense in P. We assume that the points of P_c are arranged in one sequence $\{p_i\}$ so that every point $p \in P_c$ is repeated infinitely many times, i.e. $p_i = p$, for infinitely many $i \geq 1$.

Let $\pi_P \colon W \to P$, $\pi_K \colon W \to K$ be canonical projections. Denote

$$K_i = K(p_i) = \pi_K(\pi_P^{-1}(p_i) \cap H), \tag{12}$$

$$K_i' = K_i \times \{p_i\} = \pi_P^{-1}(p_i) \cap H) \subset L \times \{p_i\}. \tag{13}$$

Since L is a G_δ-subset of K, L has a representation

$$L = \bigcap_{i \geq 1} U_i, \qquad K \supset U_1 \supset U_2 \supset U_3 \ldots$$

where $\{U_i\}$ forms (without loss of generality) a nonincreasing sequence of open subsets of K.

Fix any integer $i \leq 1$. The set $H_i = H \setminus \pi_K^{-1}(U_i) \subset W$ is closed, and so is the set $\pi_P(H_i) \subset P$ (the projection $\pi_P \colon W \to P$ is a closed map since K is compact). One observes that $p_i \notin \pi_P(H_i)$ (since $K_i \subset L \subset U_i$), and hence

$$\pi_P^{-1}(p_i) \cap H = K_i' = K_i \times \{p_i\} \subset (U_i \times \{p_i\}) \cap H \subset \pi_K^{-1} \cap H.$$

Therefore the set $V = \bigcap_{k \geq 1}(\bigcup_{i \geq k} V_i)$ is a dense G_δ-subset of P (it contains a dense subset P_c).

To complete the proof, we have to verify that $V \subset P_o$. If $v \in V$, then $v \in V_i$, for an infinite set of i. For all those i,

$$v \notin \pi_P(H_i) = \pi_P(H_i \cap \pi_K^{-1}(K \setminus U_i))$$

which implies $K(v) \subset U_i$. It follows that $K(v) \subset L = \bigcap_{i \geq 1} U_i$, and therefore $v \in P_o$. This completes the proof of Lemma A.1.

A.3. The completion of the proof

Let $\mathcal{H}$ be a separable Hilbert space. Denote by $\mathcal{L}(\mathcal{H})$ the Banach algebra of bounded linear operators on $\mathcal{H}$, and by $\mathcal{U}(\mathcal{H})$ the subset of unitary operators on $\mathcal{H}$. We recall that convergence $T_i \to T$ in the strong (or weak) operator topology means convergence $T_i f \to Tf$, for all $f \in \mathcal{H}$, in the strong (or weak, respectively) topology of the Hilbert space $\mathcal{H}$. The strong and weak operator topologies on $\mathcal{L}(\mathcal{H})$ are different, but they coincide when restricted to the set $\mathcal{U}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ (see e.g. Halmos [Ha, pp. 61–80]).

Now we view $X = S^1 \times \{0,1\}$ as a probability measure space $(X, \mathcal{B}, \mu)$ with μ a multiple of the Lebesgue measure, $d\mu = ds/2$. Take $\mathcal{H} = L^2(X, \mathcal{B}, \mu)$, and denote by $\mathcal{G}(X)$ the family of invertible measure-preserving transformations $T : X \to X$. Then $\mathcal{G}(X)$ is naturally imbedded into $\mathcal{U}(\mathcal{H})$: for $T \in \mathcal{G}(X)$ and $f \in \mathcal{H} = L^2(X, \mathcal{B}, \mu)$ define $T(f(x)) = f(T(x)) \in \mathcal{H}$. The subspace topology on $\mathcal{G}(X)$ induced by a strong (equivalently, weak) operator topology on $\mathcal{U}(\mathcal{H})$ is called weak topology on $\mathcal{G}(X)$. The following result is well known (see [Ha, p. 80]).

Lemma A.2. *The set*

$$\mathcal{E}(X) = \{T \in \mathcal{G}(X) \mid T \text{ is ergodic}\} \subset \mathcal{G}(X)$$

is a dense G_δ-subset of $\mathcal{G}(X)$ (in the weak topology of $\mathcal{G}(X)$).

Now let $K = S^1$ and $P = [0,1)$, and define $W = K \times P = S^1 \times [0,1)$, just as in Lemma A.1. The map $\phi : W \to \mathcal{G}(X)$ defined by the formula (see (7))

$$\phi(w) = \rho_w = \rho_{k,p}, \qquad \text{for } \ w = (k,p) \in W = K \times P$$

is easily verified to be continuous. In view of Lemma A.2, the set $\phi^{-1}(\mathcal{E}(X))$ is a G_δ-subset of W, and thus the complement

$$H \stackrel{\text{def}}{=} W \setminus \phi^{-1}(\mathcal{E}(X)) = \{w = (k,p) \in W \mid \phi(w) = \rho_{k,p} \text{ is not ergodic}\}$$

is an F_σ-subset of W.

Denote by $\mathbb{Q}(P)$, $\mathbb{Q}(K)$ the sets of rationals in P and $K = \mathbb{R}/\mathbb{Z} = [0,1)$, respectively. Fix an arbitrary G_δ-subset L in K such that

$$\mathbb{Q}(K) \subset L \subset K, \qquad \text{H.dim } L = 0. \tag{14}$$

To satisfy the conditions of Lemma A.1, it remains to verify that P_o is dense in P. For every $\lambda \in P$, we have $K(\lambda) = \mathrm{NE}(X_\lambda)$ (see (8), (10)), and therefore

$$P_o = \{\lambda \in P \mid \mathrm{NE}(X_\lambda) \subset L\}.$$

For $\lambda \in \mathbb{Q}(P)$, Veech [V1] proved that $\mathrm{NE}(X_\lambda) = \mathbb{Q}(K)$. Since $L \supset \mathbb{Q}(K)$ by the choice of L, the inclusion $\mathbb{Q}(P) \subset P_o$ holds. Thus P_o is indeed dense in P, and, by Lemma A.1, P_o is a residual subset of $P = [0, 1)$.

For every $\lambda \in P_o$, we have $\mathrm{NE}(X_\lambda) \subset L$; thus H.dim $\mathrm{NE}(X_\lambda) = 0$ in view of (14). This completes the proof of Theorem 5.

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